

PERIODIC ORBITS OF MECHANICAL SYSTEMS WITH HOMOGENEOUS POLYNOMIAL TERMS OF DEGREE FIVE

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ABSTRACT. In this work the existence of periodic solutions is studied for the Hamiltonian functions

$$H = \frac{1}{2} (p_X^2 + p_Y^2 + X^2 + Y^2) + \frac{a}{5} X^5 + b X^3 Y^2,$$

where the first term consist of a harmonic oscillator and the second term are homogeneous polynomials of degree 5 defined by two real parameters a and b . Using the averaging method of second order we provide the sufficient conditions on the parameters to guarantee the existence of periodic solutions for positive energy and we study the stability of these periodic solutions.

1. INTRODUCTION

The Galactic dynamics is an area of the Astrophysics where recently the application of results coming from other areas as the Celestial mechanics and the Nonlinear dynamics has gradually establish common methods and results well documented (Boccaletti and Puccaco 1999) and (Contoupoulus 2002). The global dynamics of galaxies is not a simple question and represent an actual challenge for the researches.

In the research of stellar systems, the perturbative methods of the analytical mechanics provide a simple and comprehensive description of the dynamics of systems which cannot be exactly solved with accurate quantitative predictions, even at the simplest level of the procedure. The approach used by the perturbative methods consist to study the dynamics of the original physical system by an approximating integrable system. Usually, the approximating system is an expansion in power series of the potential function in terms of the coordinates variables. There are many methods to construct the approximating integrable system as the method of the Lie transform which provides the normal form of the system which is the most simplest form that the system can be (Belmonte et al. 2008).

Many interesting problems in galactic dynamics are modeled by introducing Hamiltonian systems with two degrees of freedom X and Y of the form

$$(1.1) \quad H = \frac{1}{2} (\omega_1 p_X^2 + \omega_2 p_Y^2) + V(X, Y),$$

where ω_1 and ω_2 are the unperturbed frequencies of oscillation along the X and Y axis respectively and p_X and p_Y are the momenta conjugate to X and Y . The particular interest is to determine the properties of the orbital structure of the systems with potential functions with reflection symmetry with respect to the both axis, as examples, the potentials given by

$$(1.2) \quad V(X, Y) = \log \left(R^2 + X^2 + \frac{Y^2}{q} \right), \quad V(X, Y) = \sqrt{R^2 + X^2 + \frac{Y^2}{q}} - R^2,$$

where R is the core radius and the parameter q determines the ellipticity of the potential (Belmonte et al. 2008). In order to study the dynamics associated to the potentials (1.2)

it have been considered models where the potential function $V(X, Y)$ is expanded as a truncated series in the coordinates X and Y

$$(1.3) \quad H = \frac{1}{2} (\omega_1 p_X^2 + \omega_2 p_Y^2) + \sum_{k=0}^N \sum_{j=0}^k C_{(j, k-j)} X^j Y^{k-j},$$

where the truncation degree is N and the parameters $C_{(j, k-j)}$ are determined by the problem under study. Although the polynomial models are simplified to be considered realistic, they provide information about periodic solutions or chaos. The use of maps with polynomial models to describe the galactic motion is a useful tool because the numerical integration is faster and allows to visualize the corresponding phase space (Caranicolas and Vozikis 1999).

The polynomial Hamiltonian systems (1.3) are an actual research topic because its importance in the research of nonlinear phenomena. In models of the dynamics of galaxies the potentials with homogeneous polynomial terms of some degree N have been studied by (Caranicolas and Vozikis 2004) for the case of potential functions with homogeneous polynomials of third degree and (Contopoulos 2002) for potentials with terms of fourth degree. From the mathematical point of view, the reader can consult the following references about mechanical systems with polynomial potentials (Falconi and Lacombe 1996, 2009), (Falconi et al. 2007) and (Dorizzi and Grammaticos 1983).

There are an important cases where the frequencies are equal to one, as known 1 : 1 resonant cases. In the study of polynomial Hamiltonian systems there is an interesting problem known as the Hénon-Heiles model (Heiles and Hénon 1964), which is a simple non-integrable Hamiltonian chaotic system. The interest in the Hénon-Heiles system was initially motivated by the study of the existence of a third isolating integral of motion in certain galactic potentials admitting an axis of symmetry. The Hamiltonian proposed by Hénon-Heiles to study the existence of the third integral is given by

$$(1.4) \quad H = \frac{1}{2} (p_X^2 + p_Y^2) + \frac{1}{2} (X^2 + Y^2) + \frac{1}{3} X^3 - XY^2,$$

which is an extension of the harmonic oscillator to the anharmonic case where the perturbation term is a homogeneous polynomial of degree three. Although the model associated to Hamiltonian (1.4) is simple, the potential produce all the complexities obtainable in any chaotic system. A wide class of three particle systems can be reduced to a Hénon-Heiles type Hamiltonian by considering only the first three terms in the Taylor expansion.

Following the ideas of Hénon-Heiles it can be considered other models where the perturbation is a homogeneous polynomial of arbitrary degree with a finite number of parameters. In this paper, using the averaging method of second order, we prove for positive energy sufficiently small the existence of periodic solutions for the Hamiltonian systems consist of a harmonic oscillator plus a homogeneous potential of fifth degree with two terms and two real parameters a and b given by

$$(1.5) \quad H = \frac{1}{2} (P_X^2 + P_Y^2) + \frac{1}{2} (X^2 + Y^2) + \frac{a}{5} X^5 + bX^3Y^2.$$

We observe in the Hamiltonian systems (1.4) and (1.5) that the presence of the terms X^5 and X^3 in the expansion accounts for the breaking of the reflection symmetry with respect the X axis, the models still have the reflection symmetry with respect to the Y axis. The class of potentials studied in this paper have not chose with the aim of modeling some particular galaxies, the objective is to study systems which are generic in their basic properties.

Our main result on the periodic orbits of (1.5) is summarized in Theorem 2. The periodic orbits are the most simple non-trivial solutions of an ordinary differential system

and depending on the type of stability determine the dynamics in their neighborhood. The averaging method gives a quantitative relation between the solutions of some non autonomous differential system and the solutions of the averaged differential system, which is an autonomous one, this method leads to the existence of periodic solutions for periodic systems. The averaging is with respect to the independent variable and the right hand sides of these systems are sufficiently small, depending on a small parameter ϵ . The problem of finding periodic solutions of the perturbed differential system is reduced to find zeros of some convenient finite dimensional function. We provide the conditions under which the averaging theory guarantees the persistence of periodic orbits under the perturbation of the harmonic oscillator, and we find them as a function of the energy and the parameters a and b .

The averaging method at first and second order in the context of (Buica and Llibre 2004) has been used successfully to prove the existence of periodic solutions for the generalized Hénon-Heiles Hamiltonian system (Carrasco and Vidal 2013) and for the generalized classical Yang-Mills Hamiltonian system with two parameters (Jiménez-Lara and Llibre 2011), which consist of a perturbation of the harmonic oscillator by quartic homogeneous potentials. It is important to remark that when the potential function is of the form $V(X^2, Y^2)$ the averaging method at first order provides the information about the existence of periodic solutions, as examples, see (Jiménez-Lara and Llibre 2011) and (Llibre and Makhoul 2013). When the terms of the expansion of the potential are homogeneous polynomials of degree odd, the average system at first order vanishes in the period, hence we proceed to use the second order averaging method, as an example, see (Carrasco and Vidal 2013).

In order to get our main results this work is organized as follows. In section 2 the equations of motion are presented, a small parameter ϵ is introduced by a convenient rescaling transformation. In section 3, the results of (Buica and Llibre 2004) are presented and we apply a change of coordinates in order to use the average method of second order. In section 4, we study the conditions on the parameters a and b such that the periodic orbits exist for positive energy and we determine their stability.

2. EQUATIONS OF MOTION

The equations of motion associated to the system (1.5) are

$$(2.1) \quad \begin{aligned} \dot{X} &= P_X, \\ \dot{Y} &= P_Y, \\ \dot{P}_X &= -X - aX^4 - 3bX^2Y^2, \\ \dot{P}_Y &= -Y - 2bX^3Y. \end{aligned}$$

The Hamiltonian system (2.1) is not in the normal form for applying the averaging theory, in order to have a small parameter in the Hamiltonian system (2.1) we introduce the parameter ϵ by the change of variables (X, Y, P_X, P_Y) to (x, y, p_x, p_y) where $X = \epsilon^{\frac{1}{3}}x$, $Y = \epsilon^{\frac{1}{3}}y$, $P_X = \epsilon^{\frac{1}{3}}p_x$ and $P_Y = \epsilon^{\frac{1}{3}}p_y$ which is $\epsilon^{-\frac{2}{3}}$ -symplectic. So the system (2.1) becomes

$$(2.2) \quad \begin{aligned} \dot{x} &= p_x, \\ \dot{y} &= p_y, \\ \dot{p}_x &= -x - \epsilon(ax^4 + 3bx^2y^2), \\ \dot{p}_y &= -y - 2\epsilon bx^3y. \end{aligned}$$

The Hamiltonian function associated to the previous system is

$$(2.3) \quad H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \epsilon\left(\frac{a}{5}x^5 + bx^3y^2\right),$$

we will consider small positive values of the energy in our model. By the standard theory of Hamiltonian dynamical systems, for all $\epsilon \neq 0$ the original and the transformed systems (2.1) and (2.2) have essentially the same phase portrait. Furthermore, for $\epsilon \approx 0$ the system (2.2) is close to an integrable one.

3. SECOND ORDER AVERAGING METHOD

In this section we introduce the necessary results from averaging theory of second order for proving the statements of this paper. We follow (Buica and Llibre 2004), particularly, we focus in Theorem 3.1 that we enunciated as Theorem 1.

Theorem 1. *We consider the following differential system*

$$(3.1) \quad \dot{x} = \epsilon F_1(t, x) + \epsilon^2 F_2(t, x) + \epsilon^3 R(t, x, \epsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \rightarrow \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^n$ are continuous functions, T -periodic in the first variable, and D is an open subset of \mathbb{R}^n . We assume that

- (1) $F_1(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R and $D_x F_1$ are locally Lipschitz with respect to x , and R is differentiable with respect to ϵ . We define $f_1, f_2 : D \rightarrow \mathbb{R}^n$ as

$$(3.2) \quad \begin{aligned} f_1(z) &= \int_0^T F_1(s, z) ds, \\ f_2(z) &= \int_0^T \left(D_z F_1(s, z) \cdot \int_0^s F_1(t, z) dt + F_2(s, z) \right) ds, \end{aligned}$$

and assume moreover that

- (2) for $V \subset D$ an open and bounded set and for each $\epsilon \in (-\epsilon, \epsilon) \setminus 0$, there exists $a_\epsilon \in V$ such that $f_1(a_\epsilon) + \epsilon f_2(a_\epsilon) = 0$ and $d_B(f_1 + \epsilon f_2, V, 0) = 0$.

Then, for $|\epsilon| > 0$ sufficiently small, there exists a T -periodic solution $\varphi(\cdot, \epsilon)$ of system (3.2).

To apply Theorem 1, we introduce the following changes of variables in order to obtain a 2π -periodic system. Now, let $\mathbb{R}^+ = [0, \infty)$ and \mathbb{S}^1 the circle. We do the change of variables $(x, y, p_x, p_y) \rightarrow (r, \theta, \rho, \alpha) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1$ defined by

$$(3.3) \quad x = r \cos \theta, \quad p_x = r \sin \theta, \quad y = \rho \cos(\theta + \alpha), \quad p_y = \rho \sin(\theta + \alpha).$$

This change of variables is well defined when $r > 0$ and $\rho > 0$ and it is not canonical, so we lost the Hamiltonian structure of the differential equations.

The fixed value of the energy (2.3) in polar coordinates is

$$(3.4) \quad h = \frac{1}{2} (r^2 + \rho^2) + \epsilon \left(\frac{a}{5} r^5 \cos^5 \theta + b r^3 \rho^2 \cos^3 \theta \cos^2(\theta + \alpha) \right),$$

and the equations of motion (2.2) assume the form

$$(3.5) \quad \begin{aligned} \dot{r} &= -\epsilon \sin \theta \left[a r^4 \cos^4 \theta + 3 b r^2 \rho^2 \cos^2 \theta \cos^2(\theta + \alpha) \right], \\ \dot{\theta} &= -1 - \epsilon r \cos^3 \theta \left[a r^2 \cos^2 \theta + 3 b \rho^2 \cos^2(\theta + \alpha) \right], \\ \dot{\rho} &= -\epsilon b r^3 \rho \cos^3 \theta \sin 2(\alpha + \theta), \\ \dot{\alpha} &= \epsilon r \cos^3 \theta \left[a r^2 \cos^2 \theta + b(3\rho^2 - 2r^2) \cos^2(\alpha + \theta) \right], \end{aligned}$$

where the derivatives of the left hand side of the above equations are with respect to the time variable t , which is not periodic.

In order to write the system (3.5) as a 2π -periodic differential system we consider the angular variables θ and α . If we use the variable α as independent variable the new differential system would not have the form for applying the statements of Theorem 1.

Hence, since $\epsilon > 0$ is sufficiently small we have that $\dot{\theta} < 0$, we can change to the θ variable as the independent one. We denote by a prime the derivative with respect to θ . We obtain a new system of three differential equations by dividing system (3.5) by $\dot{\theta}$

$$\begin{aligned}
 (3.6) \quad r' &= \epsilon r^2 \sin \theta \cos^2 \theta \left[ar^2 \cos^2 \theta + 3b\rho^2 \cos^2(\alpha + \theta) \right] \\
 &\quad - \frac{\epsilon^2}{4} r^3 \sin \theta \cos^5 \theta \left[ar^2(1 + \cos 2\theta) + 3b\rho^2(1 + \cos 2(\alpha + \theta)) \right]^2 + O(\epsilon^3), \\
 \rho' &= \epsilon br^3 \rho \cos^3 \theta \sin 2(\alpha + \theta) - \epsilon^2 br^4 \rho \cos^6 \theta \sin 2(\alpha + \theta) \left[ar^2 \cos^2 \theta \right. \\
 &\quad \left. + 3b\rho^2 \cos^2(\alpha + \theta) \right] + O(\epsilon^3), \\
 \alpha' &= -\epsilon r \cos^3 \theta \left[ar^2 \cos^2 \theta + b(3\rho^2 - 2r^2) \cos^2(\alpha + \theta) \right] \\
 &\quad + \epsilon^2 r \cos^3 \theta \left[ar^2 \cos^2 \theta + b(3\rho^2 - 2r^2) \cos^2(\alpha + \theta) \right] \left[ar^3 \cos^5 \theta \right. \\
 &\quad \left. + 3br\rho^2 \cos^3 \theta \cos^2(\alpha + \theta) \right] + O(\epsilon^3).
 \end{aligned}$$

Now, the system (3.6) is 2π -periodic in the variable θ . In order to apply Theorem 1 we fix the value of the first integral at $H(r, \theta, \rho, \alpha) = h > 0$. By solving equation (3.4) for ρ we obtain

$$(3.7) \quad \rho = \sqrt{\frac{10h - 5r^2 - 2\epsilon ar^5 \cos^5 \theta}{5(1 + 2\epsilon br^3 \cos^3 \theta \cos^2(\alpha + \theta))}},$$

notice that $\rho \rightarrow \sqrt{2h - r^2}$ when $\epsilon \rightarrow 0$. Expanding (3.7) in Taylor series we obtain

$$\begin{aligned}
 (3.8) \quad \rho &= \sqrt{2h - r^2} - \frac{\epsilon r^3 \cos^3 \theta}{10\sqrt{2h - r^2}} \left[-10bh - ar^2(1 + \cos 2\theta) \right. \\
 &\quad \left. + 5br^2 + 5b(r^2 - 2h) \cos 2(\alpha + \theta) \right] + O(\epsilon^2).
 \end{aligned}$$

As we will apply averaging theory to first order, we can substitute the zero order approximation of ρ in equations (3.6), which becomes

$$\begin{aligned}
 (3.9) \quad r' &= \epsilon r \sin \theta \cos^3 \theta \left[ar^2 \cos^2 \theta + 3b(2h - r^2) \cos^2(\alpha + \theta) \right] \\
 &\quad + \frac{1}{20} \epsilon^2 r^3 \sin \theta \cos^5 \theta \left[12br^2 \cos^2(\alpha + \theta)(-10bh \right. \\
 &\quad - ar^2(1 + \cos 2\theta) + 5br^2 + 5b(r^2 - 2h) \cos 2(\alpha + \theta)) \\
 &\quad \left. - 5(6bh + ar^2(1 + \cos 2\theta) - 3br^2 + b(6h - r^2) \cos 2(\alpha + \theta))^2 \right] + O(\epsilon^3), \\
 \alpha' &= \epsilon r \cos^3 \theta \left[-ar^2 \cos^2 \theta + b(5r^2 - 6h) \cos^2(\alpha + \theta) \right] \\
 &\quad + \epsilon^2 \left[r \cos^3 \theta (ar^2 \cos^2 \theta + b(6h - 5r^2) \cos^2(\alpha + \theta)) \right. \\
 &\quad \cdot (ar^3 \cos^5 \theta - 3br(r^2 - 2h) \cos^3 \theta \cos^2(\alpha + \theta)) \\
 &\quad - \frac{3}{5} br^4 \cos^6 \theta \cos^2(\alpha + \theta)(-10bh - ar^2(1 + \cos 2\theta) + 5br^2 \\
 &\quad \left. + 5b(r^2 - 2h) \cos 2(\alpha + \theta)) \right] + O(\epsilon^3).
 \end{aligned}$$

The system (3.9) has the canonical form for applying the averaging theory of second order and satisfies the assumptions for $|\epsilon| > 0$ sufficiently small, with $T = 2\pi$ and $F_1 = (F_{11}, F_{12})$ analytical functions. Averaging the function F_1 with respect to the variable θ we obtain

$$f_1(r, \alpha) = (f_{11}(r, \alpha), f_{12}(r, \alpha)) = \int_0^{2\pi} (F_{11}, F_{12}) d\theta = (0, 0),$$

hence, the averaging theory of first order does not apply because the average functions of F_1 and F_2 vanish in the period. We proceed to calculate the function f_2 by applying the second order averaging theory. The function f_2 is defined by

$$(3.10) \quad f_2(r, \alpha) = \int_0^{2\pi} \left[D_{r\alpha} F_1(\theta, r, \alpha) \cdot y_1(\theta, r, \alpha) + F_2(\theta, r, \alpha) \right] dt,$$

where

$$y_1(\theta, r, \alpha) = \int_0^\theta F_1(t, r, \alpha) dt = \left(\int_0^\theta F_{11}(t, r, \alpha) dt, \int_0^\theta F_{12}(t, r, \alpha) dt \right).$$

The two components of the vector function y_1 are

$$(3.11) \quad \begin{aligned} y_{11} &= \frac{1}{80} r^2 \left[-16ar^2(-1 + \cos^5 \theta) \right. \\ &\quad + b(2h - r^2)(40 + 8 \cos 2\alpha - 30 \cos \theta - 10 \cos 3\theta \\ &\quad \left. - 5 \cos(2\alpha + 3\theta) - 3 \cos(2\alpha + 5\theta) - 30 \sin \theta \sin 2\alpha) \right], \\ y_{12} &= \frac{1}{240} r \left[-ar^2(150 \sin \theta + 25 \sin 3\theta + 3 \sin 5\theta) \right. \\ &\quad - b(5r^2 - 6h)(6(5 \cos \theta - 8) \sin 2\alpha) + 30 \sin \theta(2 \cos 2\alpha + 3) \\ &\quad \left. + 10 \sin 3\theta + 15 \sin(2\alpha + 3\theta) + 3 \sin(2\alpha + 5\theta) \right]. \end{aligned}$$

From Theorem 1 we arrive to the fact that the function $f_2 = (f_{21}, f_{22})$ is given by

$$(3.12) \quad \begin{aligned} f_{21} &= \frac{1}{320} br^3(r^2 - 2h) \sin 2\alpha (-240bh + (-49a + 50b)r^2 \\ &\quad + 90b(r^2 - 2h) \cos 2\alpha), \\ f_{22} &= \frac{1}{160} r^2(564b^2h^2 + 420abhr^2 - 678b^2hr^2 + 63a^2r^4 - 280abr^4 \\ &\quad + 170b^2r^4 + b(49ar^2(3h - 2r^2) + 10b(48h^2 - 51hr^2 + 10r^4)) \cos 2\alpha \\ &\quad + 45b^2(2h^2 - 3hr^2 + r^4) \cos 4\alpha). \end{aligned}$$

The components of Jacobian determinant $J(r, \alpha) = \det(D_{r,\alpha} f_2(r, \alpha))$ are

$$\begin{aligned}
 (3.13) \quad \frac{\partial f_{21}}{\partial r} &= \left[r^6 \left(\frac{7b}{320}(-49a + 50b) + \frac{63}{23}b^2 \cos 2\alpha \right) \right. \\
 &\quad - r^4 \left(\frac{15}{4}b^2 h \left(1 + \frac{3}{2} \cos 2\alpha \right) + \frac{bh}{32}(-49a + 50b) \right) \\
 &\quad \left. + \frac{9b^2 h^2 r^2}{2} \left(1 + \frac{3}{4} \cos 2\alpha \right) \right] \sin 2\alpha, \\
 \frac{\partial f_{21}}{\partial \alpha} &= r^7 \left(\frac{1}{160}(-49a + 50b) \cos 2\alpha + \frac{9}{16}b^2 \cos^2 2\alpha \right. \\
 &\quad \left. - \frac{9}{16}b^2 \sin^2 2\alpha \right) + r^5 \left(-\frac{3}{2}b^2 h \cos 2\alpha \right. \\
 &\quad \left. - \frac{1}{80}b(-49a + 50b)h \cos 2\alpha - \frac{9}{4}b^2 h \cos 4\alpha \right), \\
 \frac{\partial f_{22}}{\partial r} &= r^5 \left(\frac{189a^2}{80} - \frac{21ab}{2} + \frac{51b^2}{8} - \frac{147}{40}ab \cos 2\alpha \right. \\
 &\quad \left. + \frac{15}{4}b^2 \cos 2\alpha + \frac{27}{16}b^2 \cos 4\alpha \right) \\
 &\quad + r^3 \left(\frac{21abh}{2} - \frac{339b^2 h}{20} + \frac{147}{40}abh \cos 2\alpha \right. \\
 &\quad \left. - \frac{51}{4}b^2 h \cos 2\alpha - \frac{27}{8}b^2 h \cos 4\alpha \right) \\
 &\quad + r \left(\frac{141b^2 h^2}{20} + 6b^2 h^2 \cos 2\alpha + \frac{9}{8}b^2 h^2 \cos 4\alpha \right), \\
 \frac{\partial f_{22}}{\partial \alpha} &= \frac{1}{160} \left(r^6 ((196ab - 200b^2) \sin 2\alpha - 180b^2 \sin 4\alpha) \right. \\
 &\quad + r^4 ((-294abh + 1020b^2 h) \sin 2\alpha + 540b^2 h \sin 4\alpha) \\
 &\quad \left. + r^2 (-960b^2 h^2 \sin 2\alpha - 360b^2 h^2 \sin 4\alpha) \right).
 \end{aligned}$$

4. EXISTENCE OF PERIODIC SOLUTIONS

We seek the values of r^* and α^* such that $f_{21}(r^*, \alpha^*) = 0$, $f_{22}(r^*, \alpha^*) = 0$ and $J(r^*, \alpha^*) \neq 0$; the value of $\rho^* = \sqrt{2h - (r^*)^2}$ is obtained of (3.8) for $\epsilon = 0$. The values (r^*, α^*, ρ^*) and the energy relation (3.4) provide the initial conditions of the periodic solutions of the system (3.5) for $h > 0$ sufficiently small in coordinates $(r, \alpha, \rho, \theta)$. It is important to remark that the periodic solutions exist for a small positive level of energy, since the dynamics of our model is slightly different from the dynamics of the harmonic oscillator, when the energy increases we obtain more complex motions (Marchesiello and Puccaco 2011).

Starting with equation (3.12), if $r^* = 0, \sqrt{2h}$, or $\alpha^* = 0, \pm \frac{\pi}{2}, \pi$ then $f_{21}(r, \alpha) = 0$. Using the above values of r^* and α^* we seek the zeros of equation $f_{22}(r, \alpha) = 0$ in every case; then we evaluate the Jacobian determinant $J(r^*, \alpha^*)$. For $r^* = 0$ it easy to verify

that $J(0, \alpha) = 0$ for all α , this case is not a good solution. After calculations, we have the following proposition for the solutions (r^*, α^*, ρ^*) of the system (3.12).

Proposition 1. *For $h > 0$ small, and a, b real numbers we have that*

- (1) *If $\left| \frac{9a^2 - 10ab - 4b^2}{b(7a + 10b)} \right| < \frac{1}{2}$ and $(a - 2b)(9a - 2b)(2a - b)(2a + b)h^6 \neq 0$ there are two solutions of system (3.12) given by*

$$\left(\sqrt{2h}, \pm \frac{1}{2} \arccos \left(\frac{2(9a^2 - 10ab - 4b^2)}{b(7a + 10b)} \right), 0 \right).$$

- (2) *If $(2a + b)(-a + b) < 0$, $3b(-a + b) > 0$ and $ab^6(2a + b)(a + 3b)h^6 \neq 0$ there are two solutions of the system (3.12) given by*

$$\left(\sqrt{\frac{3bh}{-a + b}}, 0, \sqrt{-\frac{h(2a + b)}{-a + b}} \right), \quad \left(\sqrt{\frac{3bh}{-a + b}}, \pi, \sqrt{-\frac{h(2a + b)}{-a + b}} \right).$$

- (3) *If $(-a + 2b)(-a + 5b) < 0$, $b(-a + 5b) > 0$ and $(a - 10b)(a - 2b)(a + 3b)b^6h^6 \neq 0$ there are two solutions of (3.12) given by*

$$\left(\sqrt{\frac{6bh}{-a + 5b}}, 0, \sqrt{\frac{2h(-a + 2b)}{-a + 5b}} \right), \quad \left(\sqrt{\frac{6bh}{-a + 5b}}, \pi, \sqrt{\frac{2h(-a + 2b)}{-a + 5b}} \right).$$

- (4) *If $63a^2 - 182ab + 115b^2 < 0$ and $273abh - 303b^2h > 0$ or $63a^2 - 182ab + 115b^2 < 0$ and $273abh - 303b^2h < 0$ there is one solution of (3.12).*

- (5) *If $63a^2 - 182ab + 115b^2 > 0$ and $273abh - 303b^2h < 0$ there are two solutions of (3.12).*

Proof. The solutions (1)-(3) of the system are calculated directly substituting the values of $r^* = 0$, $r^* = \sqrt{2h}$ and $\alpha^* = 0$, $\alpha^* = \pi$, respectively. For $\alpha^* = \pm \frac{\pi}{2}$ we obtain the equation

$$(4.1) \quad f_{22} \left(r, \pm \frac{\pi}{2} \right) = 174b^2h^2 + (273abh - 303b^2h)r^2 + (63a^2 - 182ab + 115b^2)r^4 = 0.$$

In order to seek the positive roots of (4.1) we introduce $u = r^2$, the equation (4.1) is rewritten as

$$(4.2) \quad 174b^2h^2 + (273abh - 303b^2h)u + (63a^2 - 182ab + 115b^2)u^2 = 0.$$

The discriminant of equation (4.2) is $\Delta = 3b^2h^2(10227a^2 - 12922ab + 3923b^2)$, in order to have real solutions $\Delta \geq 0$. Using the Descartes' rule of signs the number of positive roots u^* of equation (4.1) is given in the following list

- If $63a^2 - 182ab + 115b^2 > 0$ and $273abh - 303b^2h > 0$ there are no positive roots of equation (4.2).
- If $63a^2 - 182ab + 115b^2 < 0$ and $273abh - 303b^2h > 0$ there is one positive root of equation (4.2).
- If $63a^2 - 182ab + 115b^2 < 0$ and $273abh - 303b^2h < 0$ there is one positive root of equation (4.2).
- If $63a^2 - 182ab + 115b^2 > 0$ and $273abh - 303b^2h < 0$ there are two positive roots of the (4.2).

Since we are interested in positive roots r^* of equation (4.1) we consider $r^* = \sqrt{u^*}$, hence, for each positive value of u^* there is a positive value of r^* . Now, we proceed to proof that the positive roots of (4.1) satisfy that $J(r^*, \pm \frac{\pi}{2}) \neq 0$. The Jacobian determinant evaluated at $\alpha^* = \pm \frac{\pi}{2}$ is

$$(4.3) \quad J \left(r, \pm \frac{\pi}{2} \right) = \frac{1323}{12800} (r^2 - 2h)(60bh + (7a - 20b)r^2)(6b^2h^2 + 2(3a - 7b)bhr^2 + (a - 5b)(a - b)r^4),$$

clearly, $r = \sqrt{2h}$, $\sqrt{\frac{60bh}{-7a+20b}}$ are roots of $J(r, \pm \frac{\pi}{2}) = 0$, but these values do not provide solutions of (3.12). On the other hand, the remaining roots of (4.3) are obtained from the equation

$$(4.4) \quad 6b^2h^2 + 2(3a - 7b)bhr^2 + (a - 5b)(a - b)r^4 = 0.$$

It is easy to see that the positive roots of equation (4.1) are not roots of equation (4.4), hence, if r^* is a positive root of (4.1) then $J(r^*, \pm \frac{\pi}{2}) \neq 0$. \square

In Proposition 1 the averaging method provided the initial conditions for the periodic orbits of system (2.2) in coordinates $(r, \alpha, \rho, \theta)$ which are the solutions (r^*, α^*, ρ^*) of the system (3.12) in each case. The corresponding value of θ^* is calculated directly from the energy relation (3.4) for small values of $h > 0$ using the above values of (r^*, α^*, ρ^*) . In order to characterize the periodic orbits in the original variables (x, y, p_x, p_y) we use the change of coordinates (3.3). The initial conditions for each type of periodic solution are given in the following Proposition.

Proposition 2. *For $h > 0$ positive small the initial conditions (x^*, y^*, p_x^*, p_y^*) of the periodic orbits of system (2.2) satisfy the following statements*

- (1) *If $\left| \frac{9a^2 - 10ab - 4b^2}{b(7a + 10b)} \right| < \frac{1}{2}$ and $(a - 2b)(9a - 2b)(2a - b)(2a + b)h^6 \neq 0$ we have the initial condition $(x^*, y^*, p_x^*, p_y^*) = (\sqrt{2h} \cos \theta^*, \sqrt{2h} \sin \theta^*, 0, 0)$ of one periodic orbit.*
- (2) *If $(2a + b)(-a + b) < 0$, $3b(-a + b) > 0$ and $ab^6(2a + b)(a + 3b)h^6 \neq 0$ we have the initial condition (x^*, y^*, p_x^*, p_y^*) given by*

$$\left(\sqrt{\frac{3bh}{-a+b}} \cos \theta^*, \sqrt{\frac{3bh}{-a+b}} \sin \theta^*, \sqrt{-\frac{h(2a+b)}{-a+b}} \cos \theta^*, \sqrt{-\frac{h(2a+b)}{-a+b}} \sin \theta^* \right),$$

of one periodic orbit.

- (3) *If $(-a + 2b)(-a + 5b) < 0$, $b(-a + 5b) > 0$ and $(a - 10b)(a - 2b)(a + 3b)b^6h^6 \neq 0$ we have the initial condition (x^*, y^*, p_x^*, p_y^*) given by*

$$\left(\sqrt{\frac{6bh}{-a+5b}} \cos \theta^*, \sqrt{\frac{6bh}{-a+5b}} \sin \theta^*, \sqrt{\frac{2h(-a+2b)}{-a+5b}} \cos \theta^*, \sqrt{\frac{2h(-a+2b)}{-a+5b}} \sin \theta^* \right),$$

of one periodic orbit.

- (4) *If r_0^* is a positive solution of equation (4.1) there are two initial conditions of two periodic solutions*

$$(x^*, y^*, p_x^*, p_y^*) = (r_0^* \cos \theta^*, r_0^* \sin \theta^*, -\sqrt{2h - (r_0^*)^2} \sin \theta^*, \sqrt{2h - (r_0^*)^2} \cos \theta^*),$$

$$(x^*, y^*, p_x^*, p_y^*) = (r_0^* \cos \theta^*, r_0^* \sin \theta^*, \sqrt{2h - (r_0^*)^2} \sin \theta^*, -\sqrt{2h - (r_0^*)^2} \cos \theta^*).$$

The value of θ^ is obtained from energy relation (2.2) for the initial conditions (r^*, α^*, ρ^*) and the value of the energy h .*

Proof. The initial conditions are obtained directly from the coordinate change (3.3)

$$x = r \cos \theta, \quad p_x = r \sin \theta, \quad y = \rho \cos(\theta + \alpha), \quad p_y = \rho \sin(\theta + \alpha),$$

we proof that the two solutions (r^*, α^*, ρ^*) obtained in (1), (2) and (3) in Proposition 1 are initial conditions of one periodic orbit in each case; also we proof that the conditions in (4) in Proposition 1 give two initial conditions for two periodic solutions.

- (1) For $r^* = \sqrt{2h}$, $\alpha = \pm \frac{1}{2} \arccos \left(\frac{2(9a^2 - 10ab - 4b^2)}{b(7a + 10b)} \right)$ we have that $\rho^* = 0$, then we obtained one initial condition and one periodic orbit.

- (2) For $r^* = \sqrt{\frac{3bh}{-a+b}}$ and $\rho^* = \sqrt{-\frac{h(2a+b)}{-a+b}}$, if $\alpha = 0$ and $\alpha = \pi$ we obtained two initial conditions

$$\left(\sqrt{\frac{3bh}{-a+b}} \cos \theta^*, \sqrt{\frac{3bh}{-a+b}} \sin \theta^*, \pm \sqrt{-\frac{h(2a+b)}{-a+b}} \cos \theta^*, \pm \sqrt{-\frac{h(2a+b)}{-a+b}} \sin \theta^* \right),$$

since the symmetry conditions which satisfies the variables y and p_y the above initial conditions determine the same orbit.

- (3) For $r^* = \sqrt{\frac{6bh}{-a+5b}}$ and $\rho^* = \sqrt{\frac{2h(-a+2b)}{-a+5b}}$, if $\alpha = 0$ and $\alpha = \pi$ we obtained two initial conditions

$$\left(\sqrt{\frac{6bh}{-a+5b}} \cos \theta^*, \sqrt{\frac{6bh}{-a+5b}} \sin \theta^*, \pm \sqrt{\frac{2h(-a+2b)}{-a+5b}} \cos \theta^*, \pm \sqrt{\frac{2h(-a+2b)}{-a+5b}} \sin \theta^* \right).$$

since the symmetry conditions which satisfies the variables y and p_y the above initial conditions determine the same orbit.

- (4) For $\alpha = \frac{\pi}{2}$ and $\alpha = -\frac{\pi}{2}$ the symmetry conditions of variables y and p_y are not satisfied, then we have two orbits for each r_0^* positive solution of equation (4.1). \square

According to Section 2, after a scaling with the parameter ϵ we can obtain the solutions in coordinates (X, Y, P_X, P_Y) .

The main result on the periodic orbits of the system (1.5) is summarized as follows.

Theorem 2. *For the Hamiltonian system (2.2) with small energy $h > 0$, we have the following statements*

- (1) *If $\left| \frac{9a^2 - 10ab - 4b^2}{b(7a + 20b)} \right| < \frac{1}{2}$ and $(a - 2b)(9a - 2b)(2a - b)(2a + b)h^6 \neq 0$ there is one unstable periodic solution.*
- (2) *If $(2a + b)(-a + b) < 0$, $3b(-a + b) > 0$ and $ab^6(2a + b)(a + 3b)h^6 \neq 0$ there is one unstable periodic solution.*
- (3) *If $(-a + 2b)(-a + 5b) < 0$, $b(-a + 5b) > 0$ and $(a - 10b)(a - 2b)(a + 3b)b^6h^6 \neq 0$ there is one unstable periodic solution.*
- (4) *If $63a^2 - 182ab + 115b^2 < 0$ and $273abh - 303b^2h < 0$ there are two periodic solutions.*
- (5) *If $63a^2 - 182ab + 115b^2 > 0$ and $273abh - 303b^2h < 0$ there are four periodic solutions.*

The initial conditions (x^, y^*, p_x^*, p_y^*) for each type of periodic solutions are given in Proposition 2.*

Proof. From Proposition 2, it follows number of periodic solutions of the system (1.5) in each case. The kind of stability of the periodic solutions is given by the sign of the eigenvalues λ_1 and λ_2 of the Jacobian matrix $J(r^*, \alpha^*)$ whose entries are given by (3.13). It can be verified that $J(r^*, \alpha^*) = \lambda_1 \lambda_2$ in every case. According with the hyphotesis of this theorem we have that:

For (1) the eigenvalues are real and satisfy that $\lambda_1 = -\lambda_2$, the family of periodic solutions is unstable.

For (2) the eigenvalues are

$$\lambda_1 = \frac{819}{160} \sqrt{3a(2a+b)} \left(\frac{-bh}{a-b} \right)^{\frac{7}{2}}, \quad \lambda_2 = \frac{567}{80} \sqrt{3b(a+3b)} h \left(\frac{bh}{-a+b} \right)^{\frac{3}{2}},$$

by hypothesis the parameters satisfy $b(2a+b) < 0$, there are no values of a and b such that $\lambda_1 < 0$ and $\lambda_2 < 0$, then the family of periodic solutions is unstable.

For (3) the eigenvalues are

$$\lambda_1 = \frac{189}{10} \sqrt{\frac{3}{2}} (a - 10b)(a - 2b) \left(\frac{-bh}{a - 5b} \right)^{\frac{7}{2}}, \quad \lambda_2 = -\frac{567}{20} \sqrt{\frac{3}{2}} b(a + 3b)h \left(\frac{-bh}{a - 5b} \right)^{\frac{3}{2}},$$

by hypothesis the parameters satisfy $b(2a + b) < 0$, there are no values of a and b such that λ_1 and $\lambda_2 < 0$, then the family of periodic solutions is unstable.

From $J(r, \pm \frac{\pi}{2})$ in (4.3) the eigenvalues for (4) and (5) are given by

$$\lambda = \pm \sqrt{-J(r_0^*, \pm \frac{\pi}{2})},$$

where r_0^* is given by Proposition 2, hence, depending of the values of a , b and h the orbits are stable or unstable. \square

5. CONCLUSION

The existence of periodic solutions for Hamiltonian systems with polynomial homogeneous terms of fifth degree with two real parameters a and b is established. The averaging method second order can be apply to study the existence of periodic solutions for a more general Hamiltonian system with three or four parameters

$$(5.1) \quad H = \frac{1}{2} (P_X^2 + P_Y^2) + \frac{1}{2} (X^2 + Y^2) + \frac{a}{5} X^5 + bX^3Y^2 + cX^4Y.$$

It is possible to obtain the number of periodic solutions in terms of the parameters in some cases as in the Theorem 2. The main problem arises, as shown in the proof of Proposition 1, when for certain values of α^* the polynomial equations $f_{22}(r, \alpha^*) = 0$ obtained have degree greater than six, for this reason, it can not be possible to give explicit conditions in terms three parameters.

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REFERENCES

- [1] Belmonte C., Boccaletti D., Puccaco G.: Quantitative predictions with detuned normal forms, *Celest. Mech. Dyn. Astron.* **102**, 163-176 (2008)
- [2] Boccaletti D., Puccaco G.: *Theory of Orbits, Vol. 2: Perturbative and geometric methods*, Springer-Verlag, Berlin (1999)
- [3] Buica A., Llibre J.: Averaging methods for finding periodic orbits via Brouwer degree. *Bull. Sci. Math.* **128**, 7-22 (2004)
- [4] Caranicolas N., Vozikis Ch.: Chaos in a quartic dynamical model. *Cel. Mech.* **40**, 35-49 (1987)
- [5] Caranicolas N., Vozikis Ch.: Order and chaos in galactic maps. *Astron. Astrophys.* **349**, 70-76 (1999)
- [6] Carrasco D., Vidal C.: Periodic solutions, stability and non-integrability in a generalized Hénon-Heiles Hamiltonian System. *J. of Nonlinear Physics*, **20** 199-21 (2013)
- [7] Contopoulos G.: *Order and chaos in Dynamical Astronomy*. *Astron. Astrophys. Library*. Springer: Berlin, Heidelberg, New York (2002)
- [8] Dorizzi B., Grammaticos B.: Integrability of Hamiltonians with third and fourth degree polynomial potentials. *J. Math. Phys.* **24** (9), 2289-2295 (1983)
- [9] Falconi M., Lacombe E. A.: On the dynamics of mechanical systems with the homogeneous polynomial potential $V = ax^4 + cx^2y^2$. *JDDE* **21**, 527-554 (2009)
- [10] Falconi M., Lacombe E. A.: Asymptotic behavior of escape solutions of mechanical systems with homogeneous potentials. *Cont. Math.* **198**, 181-195 (1996)
- [11] Falconi M., Lacombe E. A., Vidal C.: On the dynamics of mechanical systems with homogeneous polynomial potentials of degree 4. *Bull. Braz. Math. Soc. New Series* **38**(2), 301 (2007)
- [12] Heiles C., Hénon M.: The applicability of the third integral of motion: some numerical experiments. *Astron. J.* **69**, 73-84 (1964)
- [13] Jiménez-Lara L., Llibre J.: Periodic orbits and nonintegrability of generalized classical Yang-Mills Hamiltonian Systems, *J. Math. Phys.* **52**, 205103-14 (2011)
- [14] Llibre J., Makhlof A.: Periodic orbits of the generalized Friedmann-Robertson-Walker Hamiltonian systems, *Astrophysics Space Sci* **344**, 45-50 (2013)

- [15] Marchesiello A., G. Pucacco G.: Relevance of the 1:1 resonance in galactic dynamics, The European Physical Journal Plus **126**, article id.104 (2011)

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